Reading Project:
"The Most Complicated and Fantastic Card Trick Ever Invented"

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November 14, 2008

1 Introduction

For my reading project, I chose to review an article by Kurt Eisemann entitled "Number-Theoretic Analysis and Extensions of 'The Most Complicated and Fantastic Card Trick Ever Invented." The card trick that Eisemann describes uses a few concepts that are particularly relevant to us in Math 152: inverse permutations, modular arithmetic, and relative primality. Using these concepts, the "magic" of the trick can be unmasked for what it really is: cool mathematics.

2 The Card Trick

2.1 Setup

Assign values of 1 through 13 to the ranks Ace through King. Set aside the Ace through *Queen* of spades (12 cards) and call this the "black" deck, b. Likewise, set aside the Ace through *King* of hearts (13 cards) and call this the "red" deck, r. Order the cards in the black deck according to the following congruence relation:

$$b_i \quad \equiv \quad 2^{i-1} \pmod{13} \tag{1}$$

where b_1 corresponds to the top card in the face-down black deck, b_2 corresponds to the card beneath that, etc. The order of the cards should thus be 1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7. Cut the deck at any point, putting one half of the deck on top of the other. For example, if the cut is made at 8, we have the following sequence:

$$b = 8, 3, 6, 12, 11, 9, 5, 10, 7, 1, 2, 4 \tag{2}$$

Now, think about this as the permutation that maps $8 \rightarrow 1$, $3 \rightarrow 2$, $6 \rightarrow 3$, etc. Set up the Ace through Queen in the red deck to be the inverse of this permutation. In other words, make sure that the Ace is in position 8 ($1 \rightarrow 8$), the two is in position 3 ($2 \rightarrow 3$), etc. Append the King to the end of the deck to get the final ordering of the red deck:

$$r = 10, 11, 2, 12, 7, 3, 9, 1, 6, 8, 5, 4, 13 \tag{3}$$

Before starting the trick, familiarize yourself with what Eisemann calls the "Red k-shuffle." For some integer k, $1 \leq k \leq 12$, the k-shuffle is done by first dealing out the cards face down into k "heaps" from left to right. For example, with $k = 5$ the heaps would look like

$$\begin{array}{ccccc} 10 & 11 & 2 & 12 & 7 \\ 3 & 9 & 1 & 6 & 8 \\ 5 & 4 & 13 & & \end{array}$$

where 10 is at the bottom of heap 1, 11 is at the bottom of heap 2, etc. Take note of the heap where the last card was dealt. In this case, the last card was dealt to heap $z = 3$. The heaps now need to be re-stacked *very, very carefully.*[1] Arbitrarily pick a heap i to start with (the audience will do this when you perform the trick), and stack it on top of the $i + z \pmod{k}$th heap. Keep going in this fashion by cycling forward k heaps (modulo k) and stacking the accumulated cards on top of that heap. Continue in this fashion until all of the heaps have been re-stacked.[2] As an example, if you start collecting heaps at heap $i = 2$, the new ordering of the red deck red deck will be be

$$r' = 4, 9, 11, 8, 7, 13, 1, 2, 5, 3, 10, 6, 12$$

For now, revert the red deck back to its original form (r) and prepare to wow numerous onlookers.

2.2 Performance

You are now ready to perform the trick! Start by demonstrating the "inverse" nature of the black and red decks to the audience. You can do this by asking them to pick a number, i, between 1 and 12, and demonstrate that the ith card in the black deck, b_i, gives the *location* of card i in the red deck. In the example above, if the audience member picks $i = 5$, we show b_5 and see that it is the Jack. Then show the audience member that the 5 of hearts is in the 11th position in the red deck. With them thoroughly convinced, tell the audience that you are sick and tired of this inverse relationship, and you want to mix up the order of both of these decks. Ask them to pick a number k between 1 and 12, and perform that k-shuffle as outlined above.

Once the k-shuffle is completed, fan the red deck face up to show that the order of the cards has been changed. Cut the deck "arbitrarily" so that the King is at the end of the deck. Take note of the position of the Ace after the cut—this should be a number between 1 and 12. Next, fan the black deck face up and "arbitrarily" cut the deck so that the number corresponding to the position of the Ace in the red deck is at the top of the deck.

Point out to the audience that you have now 1) shuffled the red deck, 2) cut the red deck, and 3) cut the black deck. However, the inverse relationship between the two decks still holds! Demonstrate this to their amazement.

[1] But at the same time *very, very nonchalantly.*

[2] Since $13 = mk + z$ for some integer m, we know that we can do this until all of the cards have been picked up. This follows from the relative primality of 13 and z.

3 Cool Math

How could these two decks *still* be inversely related?! Eisemann goes into detail about this, but it really boils down to looking at the two things you do after originally setting up the decks: the k-shuffle and the cuts that you perform on the red and black decks.

3.1 The k-shuffle

What is the effect of the k-shuffle? Eisemann jumps a number of some hoops to show that the effect is actually quite simple. Without going into details, the effect can be shown through a simple example. Consider the red cards arranged in order from 1 to $p = 13$. Apply a 5-shuffle to get the following:

1	2	3	4	5
6	7	8	9	10
11	12	13		

Notice that there is a constant difference of $k = 5$ within heaps. If we pick up the 1st heap and cycle through by jumping ahead 3 heaps at a time (the last card was dealt to the third heap), we get the following sequence:

$$11, 6, 1, 9, 4, 12, 7, 2, 10, 5, 13, 8, 3$$

See a pattern? In modulo 13, there is now a constant difference of $-k$=-5=8 between each card!

3.2 Cutting the Decks

Once the cut is made on this red deck to make the King come at the end, we have the following order:

$$8, 3, 11, 6, 1, 9, 4, 12, 7, 2, 10, 5, 13$$

If we think about this sequence as representing the positions 1 through 13 instead of the cards Ace through King, then we we have simply gone through the original sequence of $r_1, r_2, ...r_{13}$ and come up with a new picked every 8th card.

Using some fancy modular arithmetic, the inverse relationship between the red deck and the black deck can be recovered by simply permuting the black deck cyclically—i.e. cutting the black deck in the correct place.

4 Conclusion

Eisemann finishes the paper by introducing an extension that can make the card trick even neater—introducing a diamond deck, shuffling the *black* deck, and *still* creating a reciprocal relationship between all three decks. The extension illustrates the same concepts that validated the original card trick, only now much more involved.

All of these serve to show the interesting nature of modular arithmetic. It's interesting to think about cutting a deck of cards as being a simple cyclic permutation of the cards. It's also interesting that the k-shuffle involves the concept of relative primality in order to guarantee that every heap is eventually picked up.